Quadrature in Ancient Egypt Revisited

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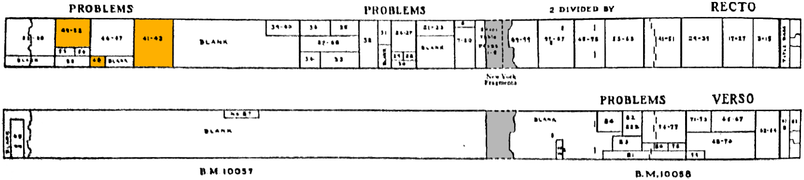
The Rhind Mathematical Papyrus

- Discovered in Thebes in (or shortly before) 1858,
- Bought by Alexander Henry Rhind in 1858,
- Acquired by British Museum in 1865,
  Under cat. no. BM 10057, BM 10058. A few fragments in the Brooklyn Museum, cat. no. 37.1784Ea-b.
- Papyrus dates from around 1542 BC,
- May be a copy of an original dating from 1840–1800 BC.

The Rhind Mathematical Papyrus



- The usual numbering of the problems by Chace & Manning's,
- Contains tables of ${}^{2}\!/_{n}$ and ${}^{n}\!/_{10}$ fractions,
- Contains arithmetic and simple “algebraic” problems,
- Contains problems concerned with areas and volumes.

Circular areas and circular-base volumes

The problems that interest us are of the form :

- A circular area of diameter d : what is its area ?
- A cylindrical volume of diameter d and height h : what is its volume ?

The Ancient Egyptians did have a method of computing these values, and in a *rather accurate* way.

Problems 41–43

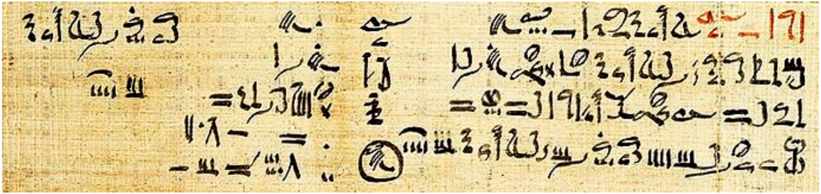
Problems concerned with cylindrical volumes.

- Problem 41 : volume of a cylindrical granary, from radius and height in cubits)
- Problem 42 : same as 41, but with unit conversion, (from cubits^3 to 'khar')
- Problem 43 : same as 41, except starting measures in khars.

They establish :

- $\left(d - {}^1\!/\!{}_9 d\right)^2 = \left({}^8\!/\!{}_9 d\right)^2$ as the area of a circle of diameter d (while the exact formula is ${}^{\pi}\!/\!{}_4 d^2$).
- $\left(d - {}^1\!/\!{}_9 d\right)^2 h$ as the volume of the cylinder.

Problem 50



- The area of a circle of diameter d is $\left(d - {}^{1}\!/_{9}d\right)^2$
- Circle reads "9 khets" (another unit of length)
- Area is $8^2 = 64$ st$\mathit{3}$t (setjat=khet^2)

Problem 50

Method to compute a circular area of 9 khets | What is the amount of its area ? | Then you subtract its $^1/_9$, resulting 1 | The remainder is 8 | Then you multiply 8 by 8. It results 64 | It is the amount of its area, 64 setjats | The procedure is

$$\begin{array}{cc} 1 & 9 \\ \overline{9} & 1 \end{array}$$

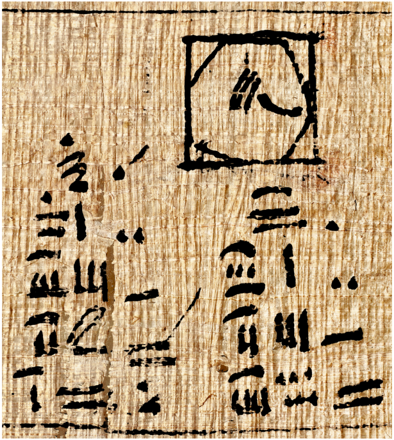
Subtract it (to 9), the remainder is 8

$$\begin{array}{rcr} & 1 & 8 \\ & 2 & 16 \\ & 4 & 32 \\ \backslash & 8 & 64 \end{array}$$

The amount is 64 setjats.

(source : Michel, Imhausen.)

Problem 48



- Shows "diagram"
- Shows two squaring procedures,
- May be the work of a different scribe, maybe an instructor.
- No problem statement.

Problem 48

No text, only the diagram and the details of two squaring :

.	8	st3t		\ .	9	st3t
2	16	st3t		2	18	st3t
4	32	st3t		4	36	st3t
\ 8	64	st3t		\ 8	72	st3t
				dmd	81	st3t

(source : Michel, Imhausen.)

Does it imply the ratio $\frac{64}{81}$ as an approximation to $\frac{\pi}{4}$?

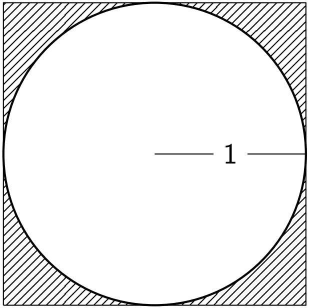
Problem 48

- The diagram is $\approx$ 15cm $\times$ 15cm
- Is the diagram explanatory ?
- If so, *what* does it show ?

A circle in a square ?

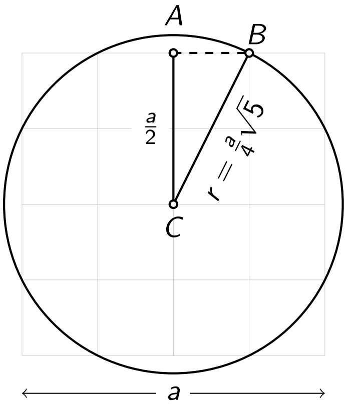
An octagon in a square ?

The *surprising* precision of the formula



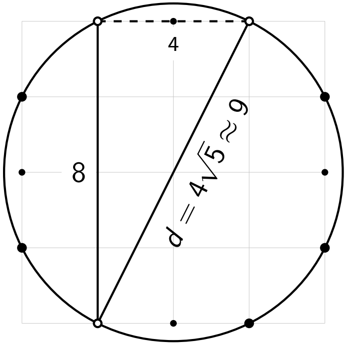
- The ratio of areas is $\frac{\pi}{4} = 0.785398\ldots$
- The Ancient Egyptians' formula gives $\frac{64}{81} = 0.79012\ldots$
- The Ancient Egyptians' approximation is $\approx 0.6\%$ off. Not bad !

According to Engels



- If $a = 8$, then $r = 2\sqrt{5} \approx 4\frac{1}{2}$, and $d \approx 9$ $(r = 4.47213\ldots\ d = 8.9442\ldots)$
- Therefore, the square and the circle have $\approx$ the same area.
- The circle has diameter ≈ 9,
- The square has area $8^2 = 64$,
- Establishing $A \approx \left({}^{8}/_{9}\,d\right)^2$.

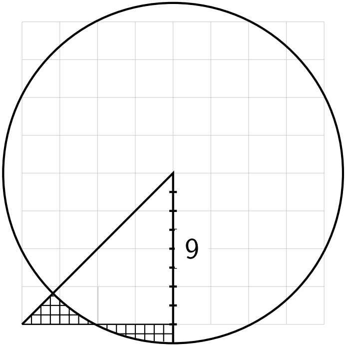
According to Robin & Shute



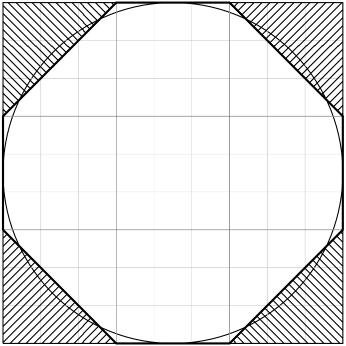
- A variation that puts the diameter directly in relation with the side of the square,
- ...but doesn't change the line of reasoning.
- ...we still get a square of area $8^2 = 64$.

A (possible) justification by Dorka

- How do we know the circle and the square areas are close ?
- Are the corners of the square outside the circle have $\approx$ same area as the circle segments outside the square.
- Dorka shows that a 18×18 grid gives the best results, with an error of ≈ 0.4 %

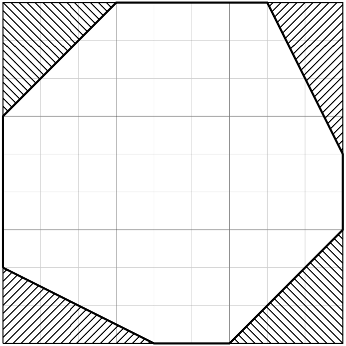


According to Vogel



- Tries to explain ,
- Proposes an irregular octagon of area 63.
- Build a square of equal area, $\sqrt{63}$. Since $\sqrt{63} \approx 8$, use 8^2.
- This explanation is accepted by Gillings.

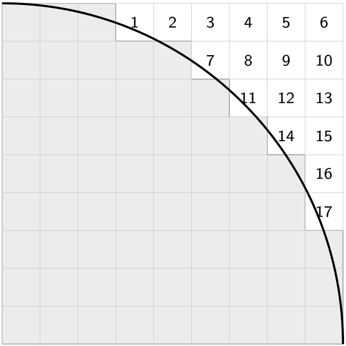
According to Guillemot



- Tries to explain ,
- The corners have area 17, the irregular octagon 64,
- Supposes the diagram is to be understood *literally*.

According to Struve & Turaev

- Tries to explain ${}^{64}/_{81}$, and why it is so precise,
- Uses a 9×9 grid,
- Finds 17 squares (mostly) outside the circle,
- Simple reasoning applies result to a whole circle.



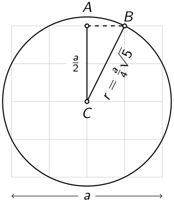
Why $(1 - {}^1/_9)^2$?

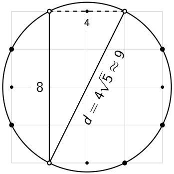
The real question remains : where does $(1 - {}^1/_9)^2$ from ?

The hypotheses are :

- Engel's “classical quadrature”
- Vogel's “hybrid quadrature”
- A number of *ad hoc* hypotheses (Guillemot, Struve & Turaev, etc.)

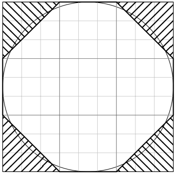
A classical quadrature ?

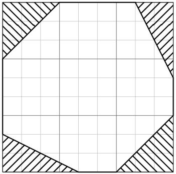




- Not justified by 
- Geometrically complicated ?
- Could they notice that $4\sqrt{5} \neq 9$?
- Need justification (cf. Dorka)

$\sqrt{63} \approx 8$ and the quadrature

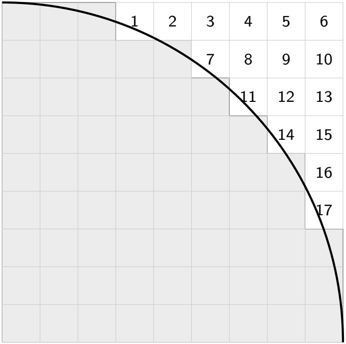




- May explain 
- Area of Vogel’s octagon is 63, so why not use the ratio ${}^7/_9$?
- Is the adjustment to 64 a quadrature, or a precision fix (and, if so, what explains it) ?

${}^{64}/_{81}$ and "mise au carreau"

- Could it be just a ratio and not a quadrature ?
- Then why express it as $(1 - {}^1/_9)^2$? Is it a computational shortcut ?
- Does not explain 



Computational complexity

- $\frac{7}{9}$ or $\frac{8}{9}$ lead to the same kind of complexity (cf. problem 42).
- $\frac{7}{9}=\frac{2}{3}+\frac{1}{9}=\frac{1}{2}+\frac{1}{6}+\frac{1}{9}$: compute d^2, then $\left(\frac{1}{2}+\frac{1}{6}+\frac{1}{9}\right)d^2$,
- or d^2, then $\left(d^2-\frac{1}{9}d^2-\frac{1}{9}d^2\right)$,
- Even if $\frac{8}{9}=\frac{2}{3}+\frac{1}{6}+\frac{1}{18}=\frac{1}{2}+\frac{1}{3}+\frac{1}{18}$, they compute $\left(d-\frac{1}{9}d\right)^2$.
- Therefore, *maybe* a complexity issue (depends on d).

Precision

If you actually know π,

- $\pi = 3.141592654\ldots$
- $\dfrac{\pi}{4} = 0.7853981634\ldots$
- $\dfrac{7}{9} = 0.\overline{7}$, about -1% off,
- $\dfrac{64}{81} = 0.7901234567\ldots$, about $+0.6\%$ off ! $(\pi \approx \dfrac{256}{81} = 3.16049\ldots)$

"Mise au carreau", or real quadrature ?

Well, we don't know :

- None of the hypotheses explain all of the evidence,
- All make at least some sense,
- The diagram  hints to a simple geometric approximation, but...
- The formula is quadrature-like.

Was $\dfrac{64}{81}$ obtained in some other way, *then* formalized as the computation of a square ?

Complexity and precision

We cannot directly invoke complexity as an explanation of the squaring :

- Even simple combinations of $\frac{1}{9}$ and d can lead to baroque computations,
- It is thought of as a general procedure : if some problems shows a convenient $d = 9$, others have $d = 10$,
- Doesn't explain  either.

Conclusion ?

- Interesting hypotheses,
- Conflicting evidence,
- All hypotheses contradict or ignore some piece of evidence,
- Very few documentary sources.

The case isn't closed !

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